1. *Let U, V be neighbourhoods of 0 in $\mathbb{R}^n$ and let $f : U \to V$ be a diffeomorphism such that $f(0) = 0$. Then show that the restriction of f to a suitable neighbourhood of 0 can be expressed as a composition of finitely many permutations and primitive diffeomorphisms.*

Solution: [2] We will find, by induction, finitely many primitive diffeomorphisms of the form

$$g_i(x) = (x_1, \cdots, x_{i-1}, x_i + \alpha_i(x), x_{i+1}, \cdots, x_n) \tag{1}$$

and a permutation q such that $f = q \circ g_n \circ \cdots \circ g_1$. Let p_i be the projections onto the i-th coordinate, $1 \leq i \leq n$. Recall that a diffeomorphism is said to be primitive if it is of the form given above, and $\frac{\partial \alpha_i(x)}{x_i} \neq -1$. Let $f_1 = f$. For each $1 \leq k \leq n-1$, we will find a permutation q_k, and a map α_k such that if we put $f_{k+1} = q_k \circ f_k \circ g_k^{-1}$, then $p_i \circ f_{k+1} = p_i, 1 \leq i \leq k$. By this, we would get that f_n is primitive and we will set $g_n = f_n$. Then we get

$$f = q_1 \circ q_2 \circ \cdots \circ q_{n-1} \circ g_n \circ \cdots \circ g_1.$$

Hence we get the result. Consider $f_k'(0)$. Since it is invertible, there exists $j \geq k$ such that $\frac{\partial (p_k \circ f_k)}{\partial x_j} \neq 0$. Take q_k to be the transposition interchanging j and k if $j \neq k$ and the identity map if $j = k$. Let $h_k = q_k \circ f_k$. Then the (k, k)-th entry of $h_k'(0) \neq 0$. Let $\alpha_k(x) = p_k \circ h_k(x) - x_k$. Let g_k be as defined as in (1). Then g_k is a local primitive diffeomorphism. Let $f_{k+1} = q_k \circ f_k \circ g_k^{-1}$. We have $p_i \circ f_k = p_i$ and $p_i \circ q_k = p_i$ for $1 \leq i \leq k-1$. Hence

$$\begin{aligned} p_i \circ f_{k+1} &= p_i \circ q_k \circ f_k \circ g_k^{-1} \\ &= p_i \circ f_k \circ g_k^{-1} \\ &= p_i \circ g_k^{-1} \\ &= p_i, 1 \leq i \leq k-1. \end{aligned}$$

For $i = k$, we get $p_k \circ f_{k+1} = p_k \circ h_k \circ g_k^{-1} = p_k \circ g_k \circ g_k^{-1} = p_k$. □

2. (a) *Let $f : U \to \mathbb{R}$ be a continuous function such that $\int_U |f(x)|\ dx < \infty$. Here U is an open set in $\mathbb{R}^n$. Let $U_1, \cdots, U_k$ be open subsets of U whose union is U. Then show that $\int_U f(x)\ dx = \sum_{\emptyset \neq I \subseteq \{1, \cdots, k\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} U_i} f(x)\ dx$.*

   (b) *Let U, V be open subsets of $\mathbb{R}^n$ and let $g : V \to U$ be a diffeomorphism. Consider the class C consisting of all open subsets U_0 of U such that, for every continuous function $f : U \to \mathbb{R}$, satisfying $\int_U |f(x)|\ dx < \infty$, the formula*

$$\int_{V_0} f(x)\ dx = \int_{g^{-1}(V_0)} |\det g'(x)| f(g(x))\ dx$$

is valid for all open subsets V_0 of U_0. Prove (without using the change of variable theorem!) that this class C is closed under finite union. You may use (with or without proof!) the result of part (a).

Solution:

(a) We will prove the result by induction. Let $k = 2$. Then $I \in \{\{1\},\{2\},\{1,2\}\}$. Clearly $\int_U f(x)\ dx = \int_{U_1} f(x)\ dx + \int_{U_2} f(x)\ dx - \int_{U_1 \cap U_2} f(x)\ dx$. Let us assume the result is true for k and prove it for $k+1$. Suppose $U = \cup_{i=1}^{k+1} U_k$. Then $\int_U f(x)\ dx = \int_{\cup_{i=1}^k U_i} f(x)\ dx + \int_{U_{k+1}} f(x)\ dx - \int_{U_{k+1} \cap \cup_{i=1}^k U_i} f(x)\ dx = \int_{\cup_{i=1}^k U_i} f(x)\ dx + \int_{U_{k+1}} f(x)\ dx - \int_{\cup_{i=1}^k U_{k+1} \cap U_i} f(x)\ dx$. Let $V_i = U_i \cap U_{k+1}$ for all $1 \le i \le k$. Then by the induction hypothesis,

$$\int_{\cup_{i=1}^k U_i} f(x)\ dx = \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} U_i} f(x)\ dx$$

and

$$\int_{\cup_{i=1}^k U_{k+1} \cap U_i} f(x)\ dx = \int_{\cup_{i=1}^k V_i} f(x)\ dx = \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} V_i} f(x)\ dx.$$

Hence

$$\begin{aligned}\int_U f(x)\ dx = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} U_i} f(x)\ dx \\ & + \int_{U_{k+1}} f(x)\ dx - \sum_{\emptyset \neq I \subseteq \{1,\cdots,k\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} U_i \cap U_{k+1}} f(x)\ dx \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,k+1\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} U_i} f(x)\ dx.\end{aligned}$$

(b) We will use part (a) to prove the result. Let $U_1, \cdots U_n \in C$ and $U = \cup_{i=1}^n U_i$. Let $V \subseteq U$ be an open subset of U. Then, if $V_i = V \cap U_i, 1 \le i \le n$, $V = \cup_{i=1}^n V_i$. Also, note that $\cap_{i=1}^n V_i$ is an open subset of U_i, for all $1 \le i \le n$. Then

$$\begin{aligned}\int_V f(x)\ dx &= \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} V_i} f(x)\ dx \\ &= \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{\#(I)-1} \int_{\cap_{i \in I} g^{-1}(V_i)} |\det g'(x)| f(g(x))\ dx \\ &= \int_V |\det g'(x)| f(g(x))\ dx.\end{aligned}$$

3. *(a) Give an example of a C^1-function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f'(x)$ is invertible for all x, but f is neither one-one nor onto.*

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by $f(x,y) = (y,x)$. Show that there is no neighbourhood of zero on which f can be written as the composition of two primitive diffeomorphisms.

Solution:

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (e^x \cos y, e^x \sin y)$. Then $f(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$. Then $f'(x,y)$ is invertible for all $(x,y) \in \mathbb{R}^2$, but f is not one-one (for example, $(0,y)$ and $(2\pi, y)$ map to the same point), and f is not onto (for example, $(0,0)$ is not in the the range of f). $\square$

(b) Suppose $f(x,y) = (y,x)$ is given on some neighbourhood of zero as the composition of two primitive diffeomorphisms, i.e., $(y,x) = (x + \alpha(x,y), y + \beta(x,y))$, for some functions α and β. Recall that for a function of the form $g(x,y) = (x + \gamma(x,y), y)$ to be a local diffeomorphism, we must have $\frac{\partial \gamma(x,y)}{\partial x} \neq -1$. Now, on the one hand, $f'(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for every $(x,y) \in \mathbb{R}^2$. On the other hand, $f'(x) = \begin{bmatrix} 1 + \frac{\partial \alpha(x,y)}{\partial x} & \frac{\partial \alpha(x,y)}{\partial y} \\ \frac{\partial \beta(x,y)}{\partial x} & 1 + \frac{\partial \beta(x,y)}{\partial y} \end{bmatrix}$. This forces $\frac{\partial \alpha(x,y)}{\partial x} = -1$, a contradiction.

4. *Define the exterior derivative $d(w)$ of a differential k-form in the class C^1. Show that if ω, λ are k-form and l-form in the class C^1, then $d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda$. Hence deduce that if ω is in the class C^2, then $dd\omega = 0$.*

Solution: Let E be an open set in $\mathbb{R}^n$. Recall that a differential k-form in E is a function ω, symbolically represented by the sum

$$\omega = \sum a_{i1,\cdots,i_k}(\mathbf{x}) dx_{i_1} \wedge \cdots dx_{i_k},$$

where the indices $i_1, \cdots, i_k$ range independently from 1 to n. The function ω assigns to each k-surface $\mathbf{\Phi}$ in E a number $\omega(\mathbf{\Phi}) = \int_D \sum a_{i1,\cdots,i_k}(\mathbf{\Phi}(\mathbf{u})) \frac{\partial(x_{i_1}, \cdots x_{ik})}{\partial(u_{i_1}, \cdots u_{ik})} d\mathbf{u}$, where $D \subseteq \mathbb{R}^k$ is the parameter domain of $\mathbf{\Phi}$. The derivative d of a k-differential form of class C^1 gives a $(k+1)$-form and is defined as follows: A 0-form of class C^1 is just a real function and we define

$$df = \sum_{i=1}^{n} (D_i f)(\mathbf{x}) dx_i.$$

Now, if $\omega = \sum b_I(\mathbf{x}) dx_I$ is the standard presentation of a k-form ω, and $b_I \in C^1(E)$ for each increasing k-index I, then we define

$$d\omega = \sum_I (db_I) \wedge dx_I.$$

For the proof of the result, see Theorem 10.20 in [1].

5. *Let $f : U \to V$ be a homeomorphism between two open subsets of $\mathbb{R}^n$. Suppose f is in the class C^k, i.e., all the kth order (and smaller order) partial derivatives $D_{i_1}, D_{i_2}, \cdots, D_{i_k} f$ exists and are continuous, for all $1 \leq i_1, \cdots, i_k \leq n$. Then show that f^{-1} is in the class C^k.*

Solution: We will prove the result by induction on k. First recall from the proof of the inverse function theorem that if $f : U \to V$ is a homeomorphism and if $f \in C^1$, then $f^{-1} \in C^1$, and further,

$$(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}. \tag{2}$$

Now suppose the result is true for $k-1$ and suppose $f \in C^k$. Then $f^{-1} \in C^{k-1}$ by the induction hypothesis. Further, $f' \in C^{k-1}$, and since the entries of the inverse of an invertible matrix are smooth functions of the entries of the matrix, $(f')^{-1}$ is also in C^{k-1}. Hence, by (2), $(f^{-1})'$ is the composition of two C^{k-1} functions, and is thus in C^{k-1}. Hence $f^{-1} \in C^k$.

References

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